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Square-free values of polynomials over the rational function field ☆

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ABSTRACT

We study representation of square-free polynomials in the polynomial ring $\mathbb{F}_q[t]$ over a finite field $\mathbb{F}_q$ by polynomials in $\mathbb{F}_q[t][x]$. This is a function field version of the well-studied problem of representing square-free integers by integer polynomials, where it is conjectured that a separable polynomial $f \in \mathbb{Z}[x]$ takes infinitely many square-free values, barring some simple exceptional cases, in fact that the integers a for which $f(a)$ is square-free have a positive density. We show that if $f(x) \in \mathbb{F}_q[t][x]$ is separable, with square-free content, of bounded degree and height, and n is fixed, then as $q \to \infty$, for almost all monic polynomials $a(t)$ of degree n, the polynomial $f(a)$ is square-free.

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1. Introduction

Let $\mathbb{F}_q$ be a finite field of q elements. We wish to study representation of square-free polynomials in the polynomial ring $\mathbb{F}_q[t]$ by polynomials in $\mathbb{F}_q[t][x]$. This is a function field version of the well-studied problem of representing square-free integers by integer polynomials, where it is conjectured that a separable polynomial (that is, without

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repeated roots) $f \in \mathbb{Z}[x]$ takes infinitely many square-free values, barring some simple exceptional cases, in fact that the integers a for which $f(a)$ is square-free have a positive density. The problem is most difficult when f is irreducible. The quadratic case was solved by Ricci [13]. For cubics, Erdös [2] showed that there are infinitely many square-free values, and Hooley [6] gave the result about positive density. Beyond that nothing seems known unconditionally for irreducible f, for instance it is still not known that $a^4 + 2$ is infinitely often square-free. Granville [3] showed that the ABC conjecture completely settles this problem. An easier problem which has recently been solved is to ask how often an irreducible polynomial $f \in \mathbb{Z}[x]$ of degree d attains values which are free of $(d-1)$-th powers, either when evaluated at integers or at primes, see [2,7–9,5,1,4,12].

In this note we study a function field version of this problem. Given a polynomial $f(x) = \sum_j \gamma_j(t)x^j \in \mathbb{F}_q[t][x]$ which is separable, that is with no repeated roots in any extension of $\mathbb{F}_q(t)$, we want to know how often is $f(a)$ square-free in $\mathbb{F}_q[t]$ as a runs over (monic) polynomials in $\mathbb{F}_q[t]$.

We want to rule out polynomials like $f(x,t) = t^2x$ for which $f(a(t),t)$ can never be square-free. To do so, recall that the content $c \in \mathbb{F}_q[t]$ of a polynomial $f \in \mathbb{F}_q[t][x]$ as above is defined as the greatest common divisor of the coefficients of f: $c = \gcd(\gamma_0, \dots, \gamma_\ell)$. A polynomial is *primitive* if $c = 1$, and any $f \in \mathbb{F}_q[t][x]$ can be written as $f = cf_0$ where f_0 is primitive. If the content c is not square-free then $f(a)$ can never be square-free.

For any field $\mathbb{F}$, let

$$\mathcal{M}_n(\mathbb{F}) = \big\{a \in \mathbb{F}[t]\colon \deg a = n,\ a \text{ monic}\big\}, \tag{1.1}$$

so that $\#\mathcal{M}_n(\mathbb{F}_q) = q^n$. Defining

$$\mathcal{S}_f(n)(\mathbb{F}) = \big\{a \in \mathcal{M}_n(\mathbb{F})\colon\ f(a) \text{ is square-free}\big\}, \tag{1.2}$$

we want to study the frequency

$$\frac{\#\mathcal{S}_f(n)(\mathbb{F}_q)}{\#\mathcal{M}_n(\mathbb{F}_q)} \tag{1.3}$$

in an appropriate limit.

There are two possible limits to take: Large degree ($n \to \infty$) while keeping the constant field $\mathbb{F}_q$ fixed, or large constant field ($q \to \infty$) while keeping n fixed. The large degree limit (q fixed, $n \to \infty$) was investigated by Ramsay [11] and Poonen [10] who showed[1] that for $f \in \mathbb{F}_q[t][x]$ separable,

$$\frac{\#\mathcal{S}_f(n)(\mathbb{F}_q)}{\#\mathcal{M}_n(\mathbb{F}_q)} = c_f + O_{f,q}\left(\frac{1}{n}\right), \quad \text{as } n \to \infty, \tag{1.4}$$

[1] They actually count all polynomials up to degree n, and do not impose the monic condition.

with

$$c_f = \prod_P \left(1 - \frac{\rho_f(P^2)}{|P|^2}\right), \tag{1.5}$$

the product over prime polynomials P, and for any polynomial $D \in \mathbb{F}_q[t]$, $\rho_f(D) = \#\{C \bmod D\colon\ f(C) = 0 \bmod D\}$. The implied constant depends on f and on the finite field size q. The density c_f is positive if and only if there is some $a \in \mathbb{F}_q[t]$ such that $f(a)$ is square-free.

In this note we deal with the large finite field limit, of $q \to \infty$ while n is fixed. Here it makes little sense to fix the polynomial f, so we also allow variable f, as long as restrict the degree (in x) and height, where for a polynomial $f(x,t) = \sum_j \gamma_j(t)x^j \in \mathbb{F}[t][x]$, the height is $\mathrm{Ht}(f) = \max_j \deg \gamma_j(t)$.

We will show

Theorem 1.1. *For all separable $f \in \mathbb{F}_q[t][x]$ with square-free content, as $q \to \infty$,*

$$\frac{\#\mathcal{S}_f(n)(\mathbb{F}_q)}{\#\mathcal{M}_n(\mathbb{F}_q)} = 1 + O\left(\frac{(n\deg f + \mathrm{Ht}(f))\deg f}{q}\right), \tag{1.6}$$

the implied constant absolute.

Thus if we fix n, the degree and the height, as $q \to \infty$ for almost all $a \in \mathcal{M}_n(\mathbb{F}_q)$ the polynomials $f(a)$ are square-free. For instance, the number of $a(t) \in \mathcal{M}_n(\mathbb{F}_q)$ for which $a(t)^4 + 2$ is square-free is, for q odd, $q^n + O(nq^{n-1})$.

Note that since primes (irreducibles) have positive density among all monic polynomials of given degree in $\mathbb{F}_q[t]$, we in particular find that for almost all primes $P \in \mathbb{F}_q[t]$ of given degree, the polynomial $f(P)$ is square-free as $q \to \infty$.

Remark. It is possible to have primitive, separable f with no square-free values, for instance take

$$f(x) = \prod_{\alpha,\beta \in \mathbb{F}_q} (x - \alpha t - \beta) = x^{q^2} + \cdots. \tag{1.7}$$

Then for all $a \in \mathbb{F}_q[t]$, $f(a)$ is divisible by $(\prod_{\gamma\in\mathbb{F}_q}(t-\gamma))^2 = (t^q - t)^2$. Indeed, if we fix $\gamma \in \mathbb{F}_q$, any $a \in \mathbb{F}_q[t]$ is congruent modulo $(t-\gamma)^2$ to some $\alpha t + \beta$ and hence $f(a) \equiv f(\alpha t + \beta) = 0 \bmod (t-\gamma)^2$. Thus we need to impose some restriction on the degree of f in Theorem 1.1.

Theorem 1.1 is a consequence of a purely algebraic result, valid over *any* field $\mathbb{F}$.

Theorem 1.2. *Suppose $f \in \mathbb{F}[t][x]$ is separable over $\mathbb{F}(t)$ and has square-free content. Then $\mathcal{S}_f(n)$ is the complement of a proper Zariski-closed hypersurface of the affine n-dimensional space $\mathcal{M}_n$, of degree $D \leqslant 2(n\deg f + \mathrm{Ht}\, f)\deg f$.*

Theorem 1.2 implies that the number of $a \in \mathcal{M}_n(\mathbb{F}_q)$ for which $f(a)$ is not square-free is at most Dq^{n-1}, where D is the total degree of an equation defining the hypersurface. Indeed, if $h \in \mathbb{F}_q[X_1, \dots, X_m]$ is a non-zero polynomial of total degree at most D, then the number of zeros of $h(X_1, \dots, X_m)$ in $\mathbb{F}_q^m$ is at most Dq^{m-1}. This is an elementary fact, seen by fixing all variables but one (cf. [14, §4, Lemma 3.1]). Hence Theorem 1.1 follows.

2. Proof of Theorem 1.2

2.1. The primitive case

We write

$$f(x,t) = \gamma_0(t) + \gamma_1(t)x + \cdots + \gamma_\ell(t)x^\ell \tag{2.1}$$

with $\gamma_j(t) \in \mathbb{F}[t]$, and $\gamma_\ell(t) \neq 0$. We first assume that $f(x,t)$ is primitive, that is $\gcd(\gamma_j(t)) = 1$. Denote by

$$\Delta_f(t) = \operatorname{disc}_x f(x,t) \tag{2.2}$$

the discriminant of $f(x)$ as a polynomial of degree ℓ with coefficients in $\mathbb{F}[t]$; it is a universal polynomial with integer coefficients in $\gamma_0(t), \dots, \gamma_\ell(t)$:

$$\Delta_f(t) = \operatorname{Poly}_{\mathbb{Z}}\big(\gamma_0(t), \dots, \gamma_\ell(t)\big) \in \mathbb{F}[t]. \tag{2.3}$$

Separability of f (over $\mathbb{F}(t)$) is equivalent to the discriminant not being the zero polynomial: $\Delta_f(t) \neq 0$.

The key observation is that $f(a) \in \mathbb{F}[t]$ being square-free is equivalent to requiring that the polynomial $t \mapsto f(a(t), t)$ does not have any multiple zeros (in any extension of the field $\mathbb{F}$). This is in fact a polynomial condition, that is a polynomial system of equations for the coefficients $a_0, a_1, \dots, a_{n-1}$ of $a(t) = a_0 + a_1 t + \cdots + a_{n-1}t^{n-1} + t^n$ which is given by the vanishing of the discriminant:

$$\operatorname{disc} f\big(a(t), t\big) = 0. \tag{2.4}$$

It suffices to show that this equation defines a *proper* hypersurface.

Before doing so, we bound the degree D of the hypersurface (2.4): For $f(x,t)$ as in (2.1), $f(a(t), t)$ is a polynomial in t of degree

$$\deg f\big(a(t), t\big) \leqslant n \deg f + \max \deg \gamma_j = n \deg f + \operatorname{Ht}(f). \tag{2.5}$$

The coefficients are polynomials in the a_j of degree at most $\deg f$. Now the discriminant of a polynomial $\sum_{j=0}^m h_j t^j$ is homogeneous in the coefficients h_j of degree $2m-2$. Hence

$a \mapsto \operatorname{disc} f(a(t), t) = \sum_k \delta_k \prod a_i^{k_i}$ has total degree at most

$$D \leqslant 2\big(n \deg f + \operatorname{Ht}(f)\big) \deg f. \tag{2.6}$$

It remains to show that Eq. (2.4) is nontrivial.

The condition that the polynomial $f(a(t))$ has multiple zeros is that there is some $\rho \in \overline{\mathbb{F}}$ (an algebraic closure of $\mathbb{F}$) with

$$f\big(a(\rho), \rho\big) = 0, \qquad \frac{\partial f}{\partial x}\big(a(\rho), \rho\big) \cdot a'(\rho) + \frac{\partial f}{\partial t}\big(a(\rho), \rho\big) = 0. \tag{2.7}$$

We define

$$W = \big\{(\rho, \vec{a}) \in \mathbb{A}^1 \times \mathbb{A}^n\colon \text{ (2.7) holds}\big\}. \tag{2.8}$$

We have a fibration of W over the ρ line $\mathbb{A}^1$ and a map $\phi : W \to \mathbb{A}^n$, the restriction of the projection $\mathbb{A}^1 \times \mathbb{A}^n \to \mathbb{A}^n$,

$$\begin{array}{ccc} & W \subset \mathbb{A}^1 \times \mathbb{A}^n & \\ {\scriptstyle \pi}\swarrow & & \searrow{\scriptstyle \phi} \\ \mathbb{A}^1 & & \mathbb{A}^n \end{array} \tag{2.9}$$

and the solutions of (2.7) are precisely $\phi(W)$.

We will show that generically the fiber $\pi^{-1}(\rho)$ has dimension $n-2$ and for at most finitely many ρ the dimension is $n-1$. Therefore we obtain that $\dim W = n-1$. Since the solutions of (2.7) are precisely $\phi(W)$, it follows that $\dim \phi(W) \leqslant n-1$. This will conclude the proof of Theorem 1.2 in the primitive case.

We note that for primitive polynomials, $f(x, \rho) = \sum_j \gamma_j(\rho) x^j$ is not the zero polynomial for any $\rho \in \overline{\mathbb{F}}$. Thus for each $\rho \in \overline{\mathbb{F}}$, the condition $f(a(\rho), \rho) = 0$ constrains a to solve an equation $a(\rho) = \beta$, where $\beta \in \overline{\mathbb{F}}$ is one of the at most $\deg f$ roots of $f(x, \rho)$.

We separate into two cases: The singular case when $\frac{\partial f}{\partial x}(a(\rho), \rho) = 0$ and the generic case when we require $\frac{\partial f}{\partial x}(a(\rho), \rho) \neq 0$.

The singular case implies that β is a multiple zero of the polynomial $f(x, \rho)$, that is that ρ is a zero of the discriminant $\Delta_f(t)$, which is not identically zero (since we assume f is separable) and hence there are only finitely many possibilities for such ρ. Given one of those ρ, then we need $a(t)$ to satisfy $a(\rho) = \beta$, i.e.

$$a_0 + a_1\rho + \cdots + a_{n-1}\rho^{n-1} + \rho^n = \beta \tag{2.10}$$

which is a (non-degenerate) linear equation, and therefore carves out an $(n-1)$-dimensional subspace of a's. Thus the singular locus consists of at most finitely many hyperplanes, and hence if non-empty has dimension $n-1$.

In the generic case, we substitute $a(\rho) = \beta$ into (2.7) to get a system

$$a(\rho) = \beta, \qquad a'(\rho) = -\frac{\frac{\partial f}{\partial t}(\beta, \rho)}{\frac{\partial f}{\partial x}(\beta, \rho)} \tag{2.11}$$

that is

$$\begin{aligned} a_0 + a_1\rho + a_2\rho^2 + \cdots + a_{n-1}\rho^{n-1} &= -\rho^n + \beta, \\ a_1 + a_2 \cdot 2\rho + \cdots + a_{n-1} \cdot (n-1)\rho^{n-2} &= -n\rho^{n-1} - \frac{\frac{\partial f}{\partial t}(\beta, \rho)}{\frac{\partial f}{\partial x}(\beta, \rho)} \end{aligned} \tag{2.12}$$

which is clearly of rank 2. Hence the fibers $\pi^{-1}(\rho)$ have dimension $n-2$.

2.2. The general case

We now relax the primitivity condition. Write $f(x,t) = c(t)f_0(x,t)$ where $f_0(x,t) = \sum_j \gamma_j^{(0)}(t)x^j$ is primitive, and $c(t) \in \mathbb{F}_q[t]$ is square-free. Since $c(t)$ is square-free, we obtain that $f(a(t),t)$ is square-free if and only if $f_0(a(t),t)$ is square-free and coprime to $c(t)$. Now $f_0(a(t),t)$ being square-free is the condition $\operatorname{disc} f_0(a(t),t) \neq 0$. For $f_0(a)$ to not be coprime to c is the algebraic condition on vanishing of the resultant

$$R = \operatorname{Res}\big(c(t), f_0\big(a(t),t\big)\big). \tag{2.13}$$

Thus the set of $a \in \mathcal{M}_n$ so that $f(a)$ is square-free is the complement of the hypersurface

$$\operatorname{disc} f_0\big(a(t),t\big) \cdot R(t) = 0. \tag{2.14}$$

We wish to show that this is a non-zero equation and to bound its total degree.

We have established above that the discriminant equation $\operatorname{disc} f_0(a(t),t) = 0$ is non-trivial, of total degree

$$D_0 \leqslant 2\big(n \deg f_0 + \operatorname{Ht}(f_0)\big) \deg f_0 = 2\big(n \deg f + \operatorname{Ht}(f_0)\big) \deg f \tag{2.15}$$

in $a_0, \ldots, a_n$.

We wish to show that the resultant R is not identically zero. Assuming (as we may) that $c(t)$ is monic, we can write the resultant as a product over the zeros of $c(t)$

$$R = \prod_{c(\alpha)=0} f_0\big(a(\alpha),\alpha\big) = \prod_{c(\alpha)=0} \sum_{j=0}^{\ell} \gamma_j^{(0)}(\alpha)\big(a_0 + \cdots + \alpha^n\big)^j. \tag{2.16}$$

For each zero α of $c(t)$, let $\ell(\alpha) = \deg f_0(x,\alpha)$ be the degree of the polynomial $f_0(x,\alpha) \in \overline{\mathbb{F}}_q[x]$, which is not the zero polynomial by primitivity of f_0. Then the total degree of R is

$$L := \sum_{c(\alpha)=0} \ell(\alpha) \leqslant \deg c \cdot \deg f \tag{2.17}$$

and the coefficient of a_0^L is $\prod_\alpha \gamma^{(0)}_{\ell(\alpha)}(\alpha)$ which is non-zero. Hence R is non-zero and of degree L.

Finally, we compute the total degree of Eq. (2.14) is the sum of D_0 and $\deg R$, which is at most

$$2\big(n \deg f + \mathrm{Ht}(f_0)\big) \deg f + \deg c \cdot \deg f \leqslant 2\big(n \deg f + \mathrm{Ht}(f)\big) \deg f \tag{2.18}$$

since $\mathrm{Ht}(f) = \mathrm{Ht}(f_0) + \deg c$. This concludes the proof of Theorem 1.2.

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